An International Journal
computers & mathematics with applications
PERGAMON Computers and Mathematics with Applications 45 (2003) 1033–1039
www.elsevier.nl/locate/camwa

Uniform Asymptotic Stability in Nonlinear Volterra Discrete Systems

P. W. Eloe, M. N. Islam and Y. N. Raffoul
Department of Mathematics
University of Dayton
Dayton, OH 45469-2316, U.S.A.
<Paul.Eloe><muhammad.islam><youssef.raffoul>@notes.udayton.edu

Abstract—We employ the notion of total stability to obtain new criteria for uniform asymptotic stability of the zero solution of a nonlinear Volterra discrete system. Resolvent equation methods are employed, and a summability criterion on the resolvent kernel is obtained. Also, we obtain a new difference equation that the resolvent $R(n, s)$ satisfies. © 2003 Elsevier Science Ltd. All rights reserved.

Keywords—Total and asymptotic stability, Resolvent, Discrete systems, Liapunov functional.

1. INTRODUCTION

We shall obtain asymptotic stability criteria for the discrete vector Volterra equation

$$x(n+1) = A(n)x(n) + \sum_{s=0}^{n} B(n,s)x(s) + g(n, x(n)), \tag{1.1}$$

for all integers $n \geq 0$ and for integers, $0 \leq s \leq n$, where $|g(n,x)| \leq \lambda(n)|x|$ for some function λ whose properties are given below.

Hino and Murakami [1,2] defined total stability for Volterra integrodifferential equations and applied the concept to obtain asymptotic stability criteria. Elaydi and Murakami [3] have extended these definitions and methods to (1.1) in the linear case with $g = 0$. Recently, Zhang [4] extended the original work of Hino and Murakami [6,7] by defining a concept, ψ-total stability. In the case $\psi = 1$, Zhang's definition reduces to that of Hino and Murakami. Zhang exhibits examples with $\psi \neq 1$ and obtains some new asymptotic stability criteria in the linear case. In each of the papers cited above, resolvent equation methods are employed and integrability or summability of associated resolvent kernels are naturally obtained. Again, Zhang obtains some new integrability results with respect to the resolvent kernel.

In this paper, we carry the definitions and methods of Zhang over to (1.1). Technically, Zhang considered the linear problem, $g = 0$. Raffoul [5] recently extended Zhang's methods to the nonlinear perturbation $g \neq 0$. We shall define ψ-total stability; 1-total stability reduces to total stability as defined by Elaydi and Murakami [3]. As a corollary, we too shall obtain new summability results for an associated resolvent kernel.

0898-1221/03/$ - see front matter Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TeX
PII: S0898-1221(03)00081-6

In addition to carrying the interesting work of Zhang [4] over to the discrete problem, we make one further contribution in this paper. In the continuous case, the resolvent, $R(t,s)$, is known to satisfy an integrodifferential equation in t; it is also known to satisfy an integrodifferential equation in s. See [6] or [4]. In the discrete case, it is known ([7] or [8]) that the resolvent, $R(n,s)$, associated with

$$x(n+1) = A(n)x(n) + \sum_{s=0}^{n} B(n,s)x(s) \tag{1.2}$$

satisfies

$$R(n+1,s) = A(n)R(n,s) + \sum_{u=s}^{n} B(n,u)R(u,s), \tag{1.3}$$

if $s \le n$, $R(s,s) = I$, and $R(n,s) = 0$ if $n < s$. In this paper, we shall also show that

$$R(n,s+1)(A(s)-I) + \sum_{u=s}^{n-1} R(n,u+1)B(u,s) + \Delta_s R(n,s) = 0, \tag{1.4}$$

if $s \le n$, $R(n,n) = I$, and $R(n,s) = 0$ if $n < s$, where $\Delta_s R(n,s) = R(n,s+1) - R(n,s)$.

Let $R = (-\infty,+\infty)$ and $Z^+ = \{0,1,2,\dots\}$. For $x \in R^k$, $|x|$ denotes the Euclidean norm of x. For any $k \times k$ matrix A, define the norm of A by $|A| = \sup\{|Ax| : |x| \le 1\}$. Let $C(n)$ denote the set of functions $\phi : [0,n] \to R^k$ and $\|\phi\| = \sup\{|\phi(s)| : 0 \le s \le n\}$. For each $\psi : Z^+ \to (0,\infty)$, we denote by $C_\psi(\tau)$ the space of all functions $p : [\tau,\infty] \to R^k$ such that $\sup_{s\ge\tau} |p(s)/\psi(s)| < \infty$. We set

$$|p|_\psi = \sup\left\{\left|\frac{p(s)}{\psi(s)}\right| : s \ge \tau\right\}.$$

For each $n_0 \in R^+$ and $\phi \in C(n_0)$, there is a unique function $x : Z^+ \to R^k$ which satisfies (1.1) on $[n_0,+\infty)$ with $x(s) = \phi(s)$ for $0 \le s \le n_0$. Such a function $x(n)$ is called a solution of (1.1) through (n_0,ϕ) and is denoted by $x(n,t_0,\phi)$.

DEFINITION 1.1. *The zero solution of (1.1) is uniformly stable (US) if for each $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that $[n_0 \ge 0,\ \phi \in C(n_0),\ \|\phi\| < \delta]$ imply $|x(n,n_0,\phi)| < \varepsilon$ for all $n \ge n_0$.*

DEFINITION 1.2. *The zero solution of (1.1) is uniformly asymptotically stable (UAS) if it is US and there exists a $\delta_0 > 0$ with the property that for each $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$ such that $[n_0 \ge 0,\ \phi \in C(n_0),\ \|\phi\| < \delta_0,\ n \ge N + n_0]$ imply $|x(n,n_0,\phi)| < \varepsilon$.*

DEFINITION 1.3. *The zero solution of (1.1) is ψ-totally stable (ψ-TS) if for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $[n_0 \ge 0,\ \phi \in C(n_0),\ p \in C_\psi(n_0),\ \|\phi\| < \delta,\ |p|_\psi < \delta]$ imply $|y(n,n_0,\phi,p)| < \varepsilon$, where $y(n) = y(n,n_0,\phi,p)$ is a solution of*

$$y(n+1) = A(n)y(n) + \sum_{s=0}^{n} B(n,s)y(s) + g(n,y(n)) + p(n), \qquad n \ge n_0, \tag{1.5}$$

such that $y(s) = \phi(s)$ for $s \in [0,n_0]$. It follows from the above definitions that the zero solution of (1.1) is ψ-TS implies it is US.

2. TOTAL AND ASYMPTOTIC STABILITY

Let $n_0 \ge 0$, $\phi \in C(n_0)$, and $x(n) = x(n,n_0,\phi)$ be a solution of (1.1). By the variation of parameters formula [7,9], we have

$$x(n) = R(n,n_0)\phi(n_0) + \sum_{s=n_0}^{n-1} R(n,s+1) \sum_{u=0}^{n_0-1} B(s,u)\phi(u), \tag{2.1}$$

where $R(n,s)$ satisfies (1.3).

LEMMA 2.1. *Suppose $x(n)$ is a solution of (1.2). If $R(n,s)$ satisfies (1.4), then $x(n)$ is given by equation (2.1).*

PROOF. Let $D(s) = A(s) - I$. Summing

$$\Delta(R(n,s)x(s)) = R(n,s+1)\Delta(x(s)) + (\Delta_s R(n,s))x(s)$$

from $s = n_0$ to $n-1$, we obtain

$$x(n) - R(n,n_0)\phi(n_0) = \sum_{s=n_0}^{n-1} \left[R(n,s+1)\Delta(x(s)) + (\Delta_s R(n,s))\, x(s)\right].$$

Thus,

$$\begin{aligned} x(n) - R(n,n_0)\phi(n_0) &= \sum_{s=n_0}^{n-1} R(n,s+1)\left[D(s)x(s) + \sum_{u=0}^{s} B(s,u)x(u)\right] + \sum_{s=n_0}^{n-1}(\Delta_s R(n,s))x(s) \\ &= \sum_{s=n_0}^{n-1} R(n,s+1)\left[D(s)x(s) + \sum_{u=0}^{n_0-1} B(s,u)\phi(u)\right. \\ &\quad \left. + \sum_{u=n_0}^{s} B(s,u)x(u)\right] + \sum_{s=n_0}^{n-1}(\Delta_s R(n,s))x(s) \\ &= \sum_{s=n_0}^{n-1} R(n,s+1)\sum_{u=0}^{n_0-1} B(s,u)\phi(u) + \sum_{s=n_0}^{n-1} R(n,s+1)D(s)x(s) \\ &\quad + \sum_{s=n_0}^{n-1}\sum_{u=n_0}^{s} R(n,s+1)B(s,u)x(u) + \sum_{s=n_0}^{n-1}(\Delta_s R(n,s))x(s). \end{aligned}$$

Interchange the order of summation to obtain

$$\begin{aligned} \sum_{s=n_0}^{n-1}\sum_{u=n_0}^{s} R(n,s+1)B(s,u)x(u) &= \sum_{u=n_0}^{n-1}\sum_{s=u}^{n-1} R(n,s+1)B(s,u)x(u) \\ &= \sum_{s=n_0}^{n-1}\sum_{u=s}^{n-1} R(n,u+1)B(u,s)x(s). \end{aligned}$$

Hence,

$$\begin{aligned} x(n) - R(n,n_0)\phi(n_0) &- \sum_{s=n_0}^{n-1} R(n,s+1)\sum_{u=0}^{n_0-1} B(s,u)\phi(u) \\ &= \sum_{s=n_0}^{n-1}\left[R(n,s+1)D(s) + \sum_{u=s}^{n-1} R(n,u+1)B(u,s) + \Delta_s R(n,s)\right] x(s). \end{aligned}$$

The summation on the right is zero according to (1.4), and hence, (2.1) is verified.

LEMMA 2.2. *If*

$$\sup_{n\ge 0} \frac{1}{1+|A(n)|}\sum_{s=0}^{n} |B(n,s)| \le L^* \tag{2.2}$$

and

$$\sup_{n\ge 1} \sum_{s=0}^{n-1} |R(n,s+1)|\,(1+|A(s)|) \le M^*, \tag{2.3}$$

for some positive constants L^* and M^*, then there exists a positive constant Q such that $|R(n,s)| \le Q$ for all $0 \le n_0 \le s \le n < \infty$.

PROOF. By solving equation (1.4) for $\Delta_s R(n,s)$, and summing it from s to $n-1$ and then changing the order of summation, we arrive at

$$\begin{aligned} R(n,s) &= I - \sum_{u=s}^{n-1} R(n,u+1)(I - A(u)) + \sum_{v=s}^{n-1}\sum_{u=v}^{n-1} R(n,u+1)B(u,v) \\ &= I - \sum_{u=s}^{n-1} R(n,u+1)(I - A(u)) + \sum_{u=s}^{n-1}\sum_{v=s}^{u} R(n,u+1)B(u,v). \end{aligned}$$

Thus,

$$\begin{aligned} |R(n,s)| &\le 1 + \sum_{u=s}^{n-1} |R(n,u+1)|\,(1+|A(u)|) \\ &\quad + \sum_{u=s+1}^{n-1} |R(n,u+1)|\,(1+|A(u)|) \sup_{\tau\ge 0} \frac{1}{1+|A(\tau)|}\sum_{v=0}^{\tau}|B(\tau,v)| \\ &\le 1 + M^* + M^*L^* =: Q. \end{aligned}$$

Later in the paper, with the aid of (1.4), we shall furnish an example in which we verify condition (2.3).

LEMMA 2.3. *If $y(n) = u(n-n_0)x(n)$ where $x(n)$ is a solution of (1.1), then $y(n)$ satisfies (1.5) with*

$$\begin{aligned} p(n) = (\Delta_n u(n-n_0))x(n+1) &+ \sum_{s=0}^{n} B(n,s)\left[u(n-n_0) - u(s-n_0)\right]x(s) \\ &+ u(n-n_0)g(n,x(n)) - g(n,u(n-n_0)x(n)). \end{aligned} \tag{2.4}$$

PROOF.

$$\begin{aligned} y(n+1) &= u(n+1-n_0)x(n+1) \\ &= (\Delta_n u(n-n_0))x(n+1) + u(n-n_0)x(n+1). \end{aligned}$$

The rest of the proof follows easily by substituting the right side of (1.1) into the second term of the above equality and then by adding and subtracting the necessary terms.

THEOREM 2.4. *Suppose (2.2),*

$$\frac{1}{1+|A(n)|}\sum_{s=0}^{n_0}|B(n,s)| \to 0 \tag{2.5}$$

as $n - n_0 \to \infty$ uniformly, and for any $\zeta = \zeta(\epsilon)$, $0 < \epsilon < 1$, there exists an $N > 0$ such that

$$\frac{\lambda(n)}{1+|A(n)|} < \zeta(\epsilon), \tag{2.6}$$

for all $n \ge N$ where ϵ is the one given below. If the zero solution of (1.1) is ψ-TS with $\psi = 1 + |A(n)|$, then it is UAS.

PROOF. By definition, the zero solution of (1.1) is ψ-TS implies it is US. Let $n_0 \in Z^+$ and $\|\varphi\| < \delta(1)$, where $\delta(\cdot)$ is the one given for the (ψ-TS) of (1.1) with $\psi(n) = 1 + |A(n)|$ for all $n \in Z^+$. Then $|x(n,n_0,\varphi)| < 1$ for all $n \ge n_0$. Now, for any $\epsilon > 0$, $0 < \epsilon < 1$, $\alpha > 0$, we set

$$u(t) = u(t,\alpha,\epsilon) = \begin{cases} \dfrac{1+2\alpha t}{1+\alpha\epsilon t}, & \text{for } t \ge 0, \\ 1, & \text{for } t < 0. \end{cases}$$

Set $y(n) = u(n-n_0)x(n)$; then $y(n)$ solves (2.1) where $p(n)$ is given by (2.4). It follows from (2.5) that for any $\eta > 0$, there exists an $S = S(\eta) > 0$ such that

$$\frac{1}{1+|A(n)|}\sum_{s=0}^{n-S(\eta)} |B(n,s)| < \eta,$$

for all $n \geq S(\eta)$. Also, $|u(n)| \leq 2/\epsilon$ (see [4]) and $|\Delta_n u(n-n_0)| \leq 2\alpha$. By (2.4), we have

$$\begin{aligned}
|p(n)| &\leq |\Delta_n u(n-n_0)| + \frac{1+|A(n)|}{(1+|A(n)|)}\sum_{s=n-S(\eta)+1}^{n} |B(n,s)||u(n-n_0)-u(s-n_0)| \\
&\quad + \frac{1+|A(n)|}{(1+|A(n)|)}\sum_{s=0}^{n-S(\eta)} |B(n,s)||u(n-n_0)-u(s-n_0)| \\
&\quad + (1+|A(n)|)\left[|u(n-n_0)||g(n,x(n))| + |g(n,u(n-n_0)x(n))|\right]\frac{1}{1+|A(n)|} \\
&\leq 2\alpha + 2\alpha L^* S(\eta)(1+|A(n)|) + 4\frac{\eta}{\epsilon}(1+|A(n)|) + \frac{4}{\epsilon}\frac{\lambda(n)}{1+|A(n)|}(1+|A(n)|) \\
&\leq \left[2\alpha + 2\alpha L^* S(\eta) + 4\frac{\eta}{\epsilon} + \frac{4\zeta}{\epsilon}\right][1+|A(n)|].
\end{aligned}$$

Take small numbers $\eta = \eta(\epsilon)$, $\alpha = \alpha(\epsilon)$, and $\zeta = \zeta(\epsilon)$ so that $\alpha(1+L^*S(\eta)) < \delta(1)/6$, $\eta < \delta(1)\epsilon/12$, and $\zeta < \delta(1)\epsilon/12$. Then, $|p(n)|_\psi < \delta(1)$. Consequently, it follows from the ψ-TS of the zero solution of (1.1) that $|y(n)| < 1$ for all $n \geq n_0 \geq 0$. Hence, if $n \geq n_0 + (1-\epsilon)/\alpha\epsilon$, we have

$$\begin{aligned}
|x(n,n_0,\varphi)| &= \frac{|y(n)|}{|u(n-n_0)|} \\
&< \frac{1}{|u(n-n_0)|} \\
&< \frac{1+\epsilon\alpha(n-n_0)}{1+2\alpha(n-n_0)} < \epsilon.
\end{aligned}$$

This completes the proof.

THEOREM 2.5. *If (2.2), (2.3), (2.5), and*

$$\sup_{n\geq 0}\sum_{s=0}^{n} \lambda(n) < \infty \tag{2.7}$$

hold, then the zero solution of (1.1) is UAS.

PROOF. We first show that the zero solution of (1.1) is ψ-TS with $\psi(n) = 1+|A(n)|$. Let $p \in C_\psi(n_0)$ and $y(n) = y(n,n_0,\varphi,p)$ be a solution of (1.5). By the variation of parameters formula (see [7,9]), we have

$$\begin{aligned}
y(n) = R(n,n_0)\phi(n_0) &+ \sum_{s=n_0}^{n-1} R(n,s+1)\sum_{u=0}^{n_0-1} B(s,u)\phi(u) \\
&+ \sum_{s=n_0}^{n-1} R(n,s+1)g(s,y(s)) + \sum_{s=n_0}^{n-1} R(n,s+1)p(s)
\end{aligned}$$

and

$$\begin{aligned}
|y(n)| &\leq \|\phi\|\left[R(n,n_0) + \sum_{s=n_0}^{n-1} |R(n,s+1)|(1+|A(n)|)\sup_{\tau\geq 0}\frac{1}{1+|A(\tau)|}\sum_{u=0}^{\tau}|B(\tau,u)|\right] \\
&\quad + |p|_\psi \sum_{s=n_0}^{n-1} R(n,s+1)(1+|A(s)|) + \sum_{s=n_0}^{n-1} |R(n,s+1)|\lambda(s)|y(s)| \\
&\leq \|\phi\|\left[Q + M^*L^*\right] + |p|_\psi M^* + Q\sum_{s=n_0}^{n-1}\lambda(s)|y(s)|.
\end{aligned}$$

Applying Gronwall's inequality, we obtain

$$|y(n)| \leq [\|\phi\| (Q + M^*L^*) + |p|_\psi M^*] \, e^{Q \sum_{s=n_0}^{n-1} \lambda(s)}.$$

This implies that the zero solution of (1.1) is ψ-TS, and therefore, by Theorem 2.4, it is UAS.

In the next example, we use Liapunov's direct method to directly verify condition (2.3). This is of special interest to us because, in difference equations, this produces a new summability criterion of the resolvent.

EXAMPLE 2.6. Consider the scalar equation

$$x(n+1) = a(n)x(n) + \sum_{s=0}^{n} b(n,s)x(s) + g(n, x(n)). \tag{2.8}$$

Suppose (2.2) and (2.5) hold for the scalar equation (2.8). Also, suppose there is a sequence $\varphi(n,s) : Z^+ \times Z^+ \to (0, \infty)$ such that

$$\Delta_s \varphi(n,s) \geq |b(n,s)| \tag{2.9}$$

and

$$-|a(n)| + K\,(1 - |b(n,n)| - \varphi(n,n)) \geq \beta, \tag{2.10}$$

where β and K are positive constants with $0 < K < 1$. If (2.7) holds, then the zero solution of (2.8) is UAS.

PROOF. Define the Liapunov functional [9] $V(s)$ on $[0, n-1]$ by

$$V(s) = |R(n,s)| + \sum_{u=s}^{n-1} \varphi(u,s)|R(n,u+1)|, \tag{2.11}$$

where $R(n,s)$ is the resolvent of (2.8) with $g = 0$, satisfying

$$R(n,s+1)a(s) + \sum_{u=s}^{n-1} R(n,u+1)b(u,s) - R(n,s) = 0.$$

Then,

$$\begin{aligned}
\Delta V(s) &= |R(n,s+1)| - |R(n,s)| \\
&\quad + \sum_{u=s+1}^{n-1} \varphi(u,s+1)|R(n,u+1)| - \sum_{u=s}^{n-1} \varphi(u,s)|R(n,u+1)| \\
&\geq (-|a(s)| + 1)|R(n,s+1)| - \sum_{u=s}^{n-1} |R(n,u+1)||b(u,s)| \\
&\quad + \sum_{u=s+1}^{n-1} \varphi(u,s+1)|R(n,u+1)| - \varphi(s,s)|R(n,s+1)| \\
&\quad - \sum_{u=s+1}^{n-1} \varphi(u,s)|R(n,u+1)| \\
&= [1 - |a(s)| - |b(s,s)| - \varphi(s,s)]\,|R(n,s+1)| \\
&\quad + \sum_{u=s+1}^{n-1} (\Delta_s \varphi(u,s) - |b(u,s)|)\,|R(n,u+1)| \\
&\geq [1 - |a(s)| - |b(s,s)| - \varphi(s,s)]\,|R(n,s+1)| \\
&= \left(-1 + \frac{1}{K}\right) |a(s)||R(n,s+1)| \\
&\quad + \frac{1}{K} [-|a(s)| + K\,(1 - |b(s,s)| - \varphi(s,s))]\,|R(n,s+1)| \\
&\geq k\,(1 + |a(s)|)\,|R(n,s+1)|,
\end{aligned} \tag{2.12}$$

where $k = \min[-1 + 1/K, \beta/K]$. Summing (2.12) from 0 to $n-1$ yields

$$k \sum_{s=0}^{n-1} (1 + |a(s)|)|R(n, s+1)| \leq V(n) - V(0) = |R(n,n)| - |R(n,0)|. \tag{2.13}$$

Thus,

$$\sup_{n\geq 0} \sum_{s=0}^{n-1} |R(n,s+1)|(1 + |a(s)|) < \frac{1}{k}. \tag{2.14}$$

By Theorem (2.5), the zero solution of (2.8) is UAS. Note that the Liapunov functional $V(s)$ defined by (2.11) is of general type. To see this, let $\varphi(u, s) = \sum_{v=0}^{s-1} |b(u, v)|$. Then φ satisfies (2.9) and condition (2.10) reduces to

$$-|a(n)| + K \left(1 - \sum_{s=0}^{n} |b(n, s)|\right) \geq \beta. \tag{2.15}$$

It is easy to see that (2.13) implies that $\sum_{s=0}^{n} |b(n, s)| < 1$, and hence, (2.2) is satisfied. We note that inequality (2.14) implies $R(n, s)$ is bounded for $1 \leq s \leq n$. Also, from inequality (2.13), $R(n, 0)$ is bounded, since $V(s)$ is increasing. We conclude that Lemma 2.2 is not needed for this example.

REFERENCES

1. Y. Hino and S. Murakami, Total stability and uniform asymptotic stability for linear Volterra equations, *J. London Math. Soc.* **43**, 305–312, (1991).
2. Y. Hino and S. Murakami, Stabilities in linear integrodifferential equations, In *Lecture Notes in Numerical and Applied Analysis, Volume 15*, pp. 31–46, Kinokuniya, Tokyo, (1996).
3. S. Elaydi and S. Murakami, Uniform asymptotic stability in linear Volterra difference equations, *Journal of Difference Equations* **3**, 203–218, (1998).
4. B. Zhang, Asymptotic criteria and integrability properties of the resolvent of Volterra and functional equations, *Funkcialaj Ekvacioj* **40**, 355–351, (1997).
5. Y. Raffoul, Total and asymptotic stability in linear Volterra integrodifferential equations with nonlinear perturbation, *Int. J. Differ. Eqn. Appl.* **6**, 19–28, (2002).
6. L.C. Becker, Stability consideration for Volterra integro-differential equations, Ph.D. Dissertation, Southern Illinois University, (1979).
7. S. Elaydi, Periodicity and stability of linear Volterra difference systems, *J. Math. Analy. Appl.* **181**, 483–492, (1994).
8. M. Islam and Y. Raffoul, Stability properties of linear Volterra integrodifferential equations with nonlinear perturbation, *Communication of Applied Analysis* (to appear).
9. Y. Raffoul, Boundedness and periodicity of Volterra systems of difference equations, *Journal of Difference Equations* **4**, 381–393, (1998).
10. P. Eloe and M. Islam, Stability properties and integrability of the resolvent of linear Volterra equations, *Tohoku Math. J.* **47**, 263–269, (1995).
11. P. Eloe, M. Islam and B. Zhang, Uniform asymptotic stability in linear Volterra integrodifferential equations with applications to delay systems, *Dynamic Systems and Applications* **9**, 331–344, (2000).
12. M. Islam and Y. Raffoul, Uniform asymptotic stability in linear Volterra difference equations, *PanAmerican Mathematical Journal* **11**, 61–73, (2001).